A MARXIAN MATHEMATICAL MODEL OF CAPITAL ACCUMULATION IN A TWO-SECTOR ECONOMY

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Abstract: The purpose of this paper is to construct a macroeconomic model of capital accumulation of a two-sector economy, building upon the Marxian model of expanded reproduction with which the factors of economic crises can be derived. Its methodology involved the simple generalization and formalization of the relations between the two economic sectors laid out by Marx in *Capital*, Vol. 2. On top of that, a time variable was added to make the system dynamic and so that it would evolve over time. The model has succeeded in covering some of the gaps in both Goodwin's and Foley's models by accounting for various parameters which Foley and Goodwin assumed to be constant. After the model was developed mathematically, a sensitivity analysis was run on it in which technical changes, wage fluctuations and other variables were introduced. Not only was the model found to be consistent with Marxist theory, but four causes of crises were also found following a sensitivity analysis. The model has its limitations, which are discussed in the conclusion, yet it can certainly be built upon in the future.

Keywords: capital accumulation; sector; technical change; cycles; mathematical model

JEL codes: E11; E23; E22; O11; O33

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Introduction

The two main Marxian models of capital accumulation, around which the literature in that field revolves, are Goodwin's and Foley's models, which are discussed in more depth in the literature review section. Even with contemporary expansions on this pair of models, they still lack important parameters which must be accounted for. This paper develops a macroeconomic model which builds upon Marx's model of expanded reproduction provided in *Capital*, Vol. 2, while also accounting for technical changes, wage fluctuations, the depreciation of constant capital and fluctuations in unproductive expenditures by capitalists. The model assumes an economy comprised of two sectors, allowing us to observe the relations and dynamics occurring between different forms of capital over time.

Literature Review

I shall start the theoretical background of Marxian mathematical models of capital accumulation with Goodwin's growth cycle (Choi, 1995; Goodwin, 1967).

Goodwin's main thesis is that the distribution share (of wealth) of the proletariat, the working class of society, is stable in the long term. His accumulation model assumes a single-sector economy without technical changes, i.e., fixed coefficients of production, and wages being determined by the employment rate. Based on these assumptions, his model provides results describing cyclical behaviour in capital accumulation, naturally making the growth rate of capital cyclical, as well. These cyclical patterns are said to be dependent on the fluctuating income distribution between workers and capitalists, which itself is the result of an increasing or decreasing employment rate. Fluctuations in capital accumulation continue until a stable point is reached, that is, until the employment rate becomes stable and so does the income distribution. The main cause of cycles in capital accumulation in Goodwin's model is the employment rate, while technical change doesn't have anything to do with it; that is why he assumes it to be constant.

The model has also taken other factors into consideration, such as that of monetary and financial aspects (Desai, 1975); later, technical change was introduced (Shah and Desai, 1981), as well as the possibility of both chaotic and stable growth (Pohjola, 1981). Another important contribution to Goodwin's model was the introduction of two sectors (Sato, 1985): one producing capital goods; the other, consumption goods.

Other contributions to the Marxian mathematical formalization of capital accumulation were made by Duncan Foley (1986), closely examining the circuit of capital in his analysis, i.e., the transformation of capital from one type to another, which can be expressed as: $M \rightarrow C <_{LP}^{MoP} \ldots Pr \ldots C' \rightarrow M'$, in which the variables M, C, MoP, LP, Pr represent money, commodities, means of production, labour power and production, respectively. The forms of capital in his analysis are therefore: money, productive capital and commodity capital. Foley assumes a two-sector economy, those being the sector of capital goods and the sector of consumption goods, following Marx's idea of basic and advanced reproduction from *Capital*, Vol. II (Marx, 1954). In his model, Foley considers possible changes in the value of money, defined as the ratio between labour hours and its monetary representation, bank policies, lines of credit and interest. What mainly distinguishes his model from Goodwin's is the fact that capital is analysed in all its forms and the economy is not assumed to be one-sectored. One of the things they have in common is an assumption of no technical change, which Foley makes in the "Simple and Expanded Reproduction" section by saying:

"Analytically the assumption of simple and expanded reproduction requires parameters p, q and the lag functions of the model set out above be constant through time" (Foley, 1986, p. 15).

To clarify, p, q and the lag functions represent the proportion of surplus recommitted to production, the rate of profit and any time lags due to production, the realization of surplus or a recommitment of capital (money) into production, respectively. I am assuming no technical change to mean no change in the rate of profit and no change in the time lags.

Formalisation of a Two-Sector Economy

As I have already mentioned in the introduction, my model will assume two sectors: those of capital goods and consumer goods. Before delving into the model, I will lay out the relations between Sector I and Sector II and the equations governing those relations. The initial illustration of the relations between the two will be done through the help of a numerical example, provided by Marx in *Capital*, Vol. 2, Chapter 21, which you can see in Table 1 and Table 2.

Table 1. Table of the economy before investment

	C	V	S
Sector I	$4{,}000_c$	$1{,}000_v$	$1{,}000_s$
Sector II	$1{,}500_c$	750_v	750_s

Source: Marx, 1954, Ch. 21, pp. 508–510.

Table 2. Table of the economy after investment

	C	V	S
Sector I	$4{,}400_c$	$1{,}100_v$	$1{,}100_s$
Sector II	$1{,}650_c$	825_v	825_s

Source: Marx, 1954, Ch. 21, pp. 508–510.

The numbers themselves don't have much meaning unless properly explained. The first thing I'm going to clarify is the subscripts, which are there to simply indicate which component (C, V or S) the value expressed belongs to. The numbers can refer to any unit, for example, a monetary expression of "$$" or labour hours. The components C, V and S represent constant capital (means of production, such as machines and raw materials), variable capital (value used to purchase labour power) and surplus value (which might also be called profit), respectively.

Moving on to the values of each component of each sector, it can be said that either the C or V value in Sector I was arbitrarily picked. I'm highlighting the "or" because if one of them were picked, the other could easily be calculated, assuming we know the organic composition of capital at that time – the latter being defined as C/V, which in our example is equal to $4/1$ and $2/1$, respectively, for each sector. That leaves us with the value of S, surplus value, which Marx calculates using the rate of surplus value, defined as S/V, in this case being 100%.

Having the components of Sector I clearly defined, we can move on to Sector II, which depends on the former. The constant capital of Sector II is defined to be the sum of the variable capital and the surplus value being unproductively consumed from Sector I. That is because once Sector I sells its aggregate product (6,000 in Table 1), the amount of constant capital (4,000) will be rebought by Sector I itself, since we can assume this is needed to produce the same amount as before. Then we're left with 2,000, of which 500 will stay and not be used, this for the purpose of capital accumulation, while the other 50% of the surplus value plus the variable capital (the sum of workers' wages) will presumably be spent on means of consumption and not saved, i.e., they will go to Sector II, specifically into the constant capital component of Sector II.

Knowing the constant capital of Sector II, we can calculate its variable capital using the previously defined organic composition of capital, technically using its inverse. Marx then uses the rate of surplus value to calculate the surplus value.

Table 2 represents the exact same thing but with the difference that an increase in the variable and constant capital of Sector I has occurred. Specifically, the allocation of the saved surplus value is 400 to constant capital and 100 to variable capital. Consequently, the values of all other components for both sectors change.

After this numerical example, I believe I can present a proper algebraic formalization of the values for each component of each sector in this economy (below). I will assume V_1 to be the arbitrarily chosen value, which will be used to calculate the rest. The subscript denotes the sector which this value belongs to.

$$C_1 = c_1 V_1 \tag{1}$$

$$S_1 = s_1 V_1 \tag{2}$$

$$C_2 = V_1 + a_1 S_1 \tag{3}$$

$$V_2 = c_2^{-1} C_2 \tag{4}$$

$$S_2 = s_2 V_2 \tag{5}$$

In which:

c is the organic composition of capital;

s is the rate of surplus value;

a is the portion of surplus being unproductively consumed (spent on consumer goods).

The equations governing the values in Table 2 would be the same, except that the value from which everything else is derived (either V_1 or C_1) would experience an increase.

The Model

Having clearly described the relationship between the two sectors in the economy, I shall now start to define a dynamical model, by which I mean one that observes the economy with respect to time. Before doing that, I will state the parameters to be used in the model (Table 3).

Table 3. Parameters of the model

Notation	**Description**	**Unit**
K_i	Capital stock of ith sector	Labour hours
C_i	Constant capital of ith sector	Labour hours
V_i	Variable capital of ith sector	Labour hours
S_i	Surplus value of ith sector	Labour hours
Q_i	Output value of ith sector	Labour hours
L	Hours of labour done by a worker per work period (year, week, day)	Labour hours per work period
W	Wage of a worker per work period	Labour hours per work period
c_i	Organic comp. of capital of ith sector	Dimensionless
a_i	Portion of surplus used for consumer goods in ith sector	Dimensionless
δ	Depreciation rate of constant capital	Dimensionless
t	Work period	Work period

Source: Prepared by the author.

To clarify for those who are unfamiliar with Marx's theory of value, Marxists define value as the average labour time required to produce a certain good. That is why the unit of measurement for capital stock, as well as other such goods, is stated in labour hours.

Now that we have defined its parameters, we can move on to the model.

$$Q_1(t) = \delta C_1(t) + V_1(t) + S_1(t) \quad (6)$$

Starting with the value of the output, there's almost nothing that hasn't been mentioned already in the numerical example, apart from depreciation rates, so I will clarify. I've added a depreciation rate of constant capital because I'm assuming that not all of it will be consumed in the process of production, i.e., I'm accounting for fixed capital. The constant capital from the example given in Tables 1 and 2 was the depreciated capital only, that is, the part of the constant capital within the capital stock whose value has been transferred to the produced commodities or output.

To prevent any ambiguities about output being a function of work periods, I'm going to state this clearly. $Q_1(t)$ represents the total output of Sector I produced during a work period, t. This will be the case for all other functions of t.

Moving on, I will expand on Equation 6 by assuming that the sum of the variable capital and the surplus value is the total living labour put into the production process, living labour being the total amount of labour performed by the workers.

$$Q_1(t) = \delta C_1(t) + V_1(t)\frac{L}{W} \quad (7)$$

The total living labour, as demonstrated in Equation 7, is taken by multiplying the variable capital by the ratio between total labour done by a worker per work period and the wage of that worker per work period (which itself is a pure number), resulting in a scaled version of variable capital. Usually, this ratio must be > 1 for the value of the output produced to be bigger than the actual costs, that is, so that one can have surplus value.

$$S_1(t) = Q_1(t) - \delta C_1(t) - V_1(t) = V_1(t)\frac{L}{W} - V_1(t) \quad (8)$$

The argument made above Equation 8 is clearly correct, as the equation describes the amount of surplus value. If the ratio L/W is ≤ 1, the surplus value will be either negative or 0. The only thing left now is the capital stock of that sector.

$$K_1(t) = C_1(t) + V_1(t) = V_1(t)(c_1 + 1) \quad (9)$$

Having everything we need defined, we can move on to the change of this system with respect to work periods, which we will model with a differential equation.

$$\frac{dK_1(t)}{dt} = S_1(t)(1 - a_1) = V_1(t)\left(\frac{L}{W} - 1\right)(1 - a_1) = \frac{K_1(t)}{c_1 + 1}\left(\frac{L}{W} - 1\right)(1 - a_1) \quad (10)$$

By the latter (Equation 10), I'm implying that the change in the capital stock is fully dependent on the surplus value that was not used for the purchasing of consumer goods. By expanding the initial term, I can conclude that the change in capital stock is dependent on the value of the capital stock during the current work period, the organic composition of capital, the ratio between overall labour and wage and the portion of surplus being unproductively consumed by the capitalists. I will define a change in c_1 as a technical one, as this would mean that the ratio between machines and labourers has changed, which is usually a characteristic of technical change in modern industry. One can clearly see that increases in c_1 (positive technical change) will lead to a decrease in the capital stock, which implies that the rate of profit, defined as the growth rate of capital, has decreased. I am mentioning this to show that my model is consistent with Marxist theory.

The differential equation that I have just produced could be solved analytically, but that would require an analytical definition of L, W, c_1 and a_1; of course, this could be provided, but doing so limits the scenarios which can be simulated with the model. Later in this paper, I will provide its numerical solution instead.

Having defined the dynamics of Sector I, we can proceed with Sector II.

$$C_2(t) = a_1 S_1(t) + V_1(t) \tag{11}$$

$$Q_2(t) = \delta C_2(t) + V_2(t) + S_2(t) = \delta\big(a_1 S_1(t) + V_1(t)\big) + V_2(t) + S_2(t)$$
$$= \delta\big(a_1 S_1(t) + V_1(t)\big) + V_2(t)\frac{L}{W} \tag{12}$$

$$S_2(t) = Q_2(t) - \delta\big(a_1 S_1(t) + V_1(t)\big) - V_2(t) = V_2(t)\frac{L}{W} - V_2(t) \tag{13}$$

$$K_2(t) = C_2(t) + V_2(t) = a_1 S_1(t) + V_1(t) + V_2(t) = (a_1 S_1(t) + V_1(t))(1 + c_2^{-1}) \tag{14}$$

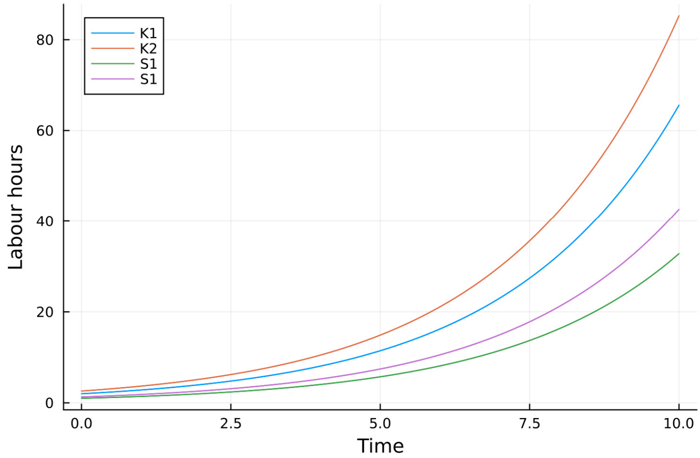
In this sector, I do not need to define the rate of change of the capital stock, since it would be determined by components from Sector I, which themselves will be determined by the capital stock of Sector I, for which I already have a differential equation that describes its evolution in terms of value over work periods.

Numerical Solution of the System

Using the programming language Julia (Bezanson et al., 2017), I have numerically computed the evolution of the system over work periods and under different conditions. The first simulation (see Appendix A) assumed the following conditions:

$a_1 = 0.3;\ c_1 = 1; c_2 = 1, W = 1; L = 2; \delta = 0.8;\ V_1(0) = 1,$ with $t \in [0,10]$ and $dt = \frac{10}{1000}$.

The results can be seen in Figure 1, which plots the surplus value and capital stock of both sectors with respect to work periods. Naturally, the growth of both is exponential. It can also be noted that the capital stock, and with it the surplus value, of Sector II appears to always be greater than that of Sector I under these conditions.

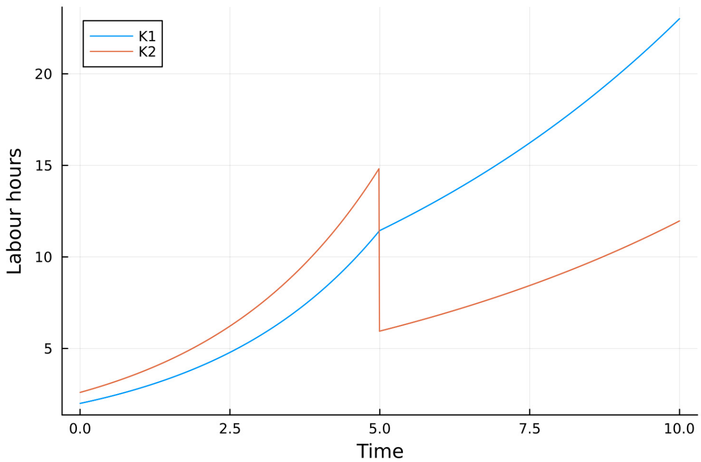


Source: Author's calculation.

Figure 1. Surplus value and capital stock with respect to time

Introduction of Technical Change

I shall now observe the system's evolution, assuming a technical change in both sectors. I will start with a technical change in only Sector I that can be expressed as an increase of c_1 during a specific work period, which I will choose to be 5 in this case. I will avoid plotting surplus values this time because they're nothing but a scaled version of capital stock. The results are plotted in Figure 2 (see Appendix B for the code).



Source: Author's calculation.

Figure 2. Introduced technical change in Sector 1

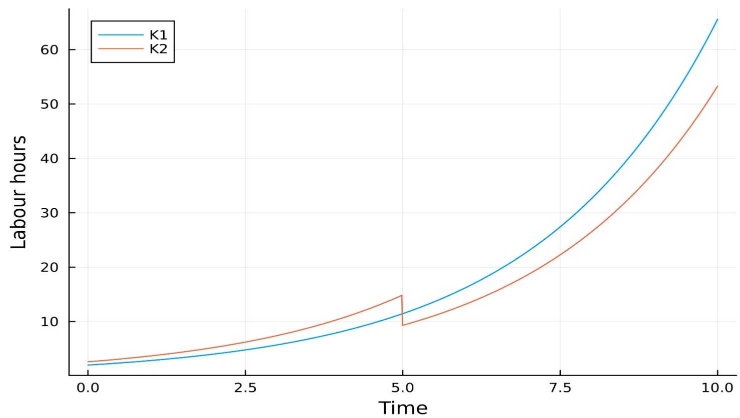
$$c_1(t) = 1, \forall t \in [0,5]; c_1(t) = 4, \forall t \in (5,10]$$

Looking at Figure 2, we can notice a few things. For starters, after the technical change introduced at point $t = 5$, the derivative of the capital stock of Sector I (its slope) reduced, which was expected as we have already noted below Equation 10. Observing the capital stock of Sector II, we see a sudden drop in the capital stock. Why does that happen? This sudden drop is caused by the change in the composition of the capital stock of Sector I. Due to the fact that K_1 is defined as:

$$K_1(t) = V_1(t)(c_1(t) + 1),$$

which we have already stated in Equation 9, once c_1 increases, another term must decrease in order for the capital stock to preserve its value. That term is $V_1(t)$. Once it decreases, so does the demand for consumer goods. Finally, in order for Sector II to decrease its output, due to the decrease in demand, the overall capital stock that drives production must also be reduced. If sudden shocks in the organic composition of capital in Sector II continue to happen, it is believed that the capital stock of Sector II will follow a cyclical pattern.

The next thing that we will simulate is a technical change in Sector II, while c_1 stays constant. The increase in c_2 will be the same as the one in c_1 from the previous example at that same time, $t = 5$. The results can be seen in Figure 3.



Source: Author's calculation.

Figure 3. Introduced technical change in Sector II

$$c_2(t) = 1, \forall t \in [0,5]; c_2(t) = 4, \forall t \in (5,10]$$

As can be observed, there is a sudden drop in only the capital stock of Sector II, while Sector I's capital stock continues to increase uninterruptedly. The cause behind this can be found in Equation 14:

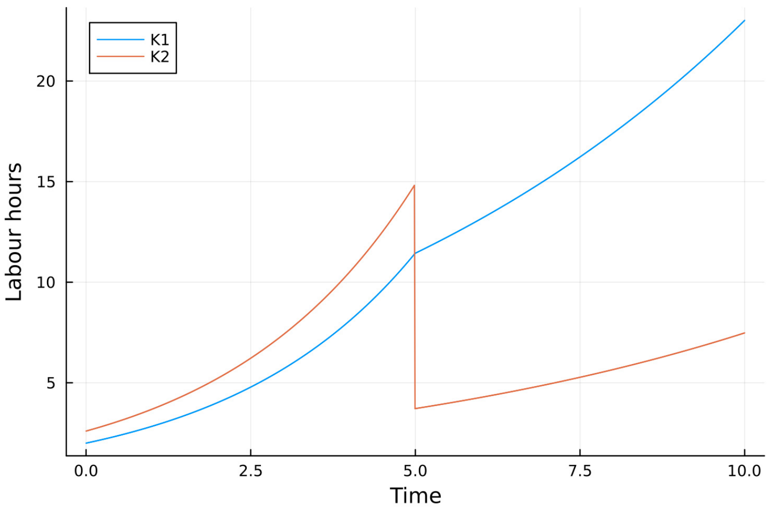
$$K_2 = (a_1 S_1 + V_1)(1 + c_2^{-1}).$$

The increase in c_2 is nothing but a decrease of c_2^{-1}, which is its inverse. As a result, the capital stock is reduced overall, more specifically the variable capital component.

Finally, if we introduce a technical change in both sectors at the same time, the following result will occur (Figure 4).

One can notice that this is nearly identical to Figure 2, with the difference being that there is a sharper decrease here in the capital stock of Sector II. This is completely expected and can be explained by the combination of the two previous explanations.

If we assume that more of these technical changes continue to occur at given points in time, which will indeed happen due to competition in the capitalist market, it is certain that the capital stock of Sector II will undergo a similar cycle, regardless of which of the two sectors the technical change takes place in.



Source: Author's calculation.

Figure 4. Technical change in both sectors

Introduction of a Non-Constant Wage

Till now, we have been assuming throughout all simulations that the wage per worker and per work period is constant. Of course, this is not always the case in a real-world situation. Wages fluctuate due to several factors, such as labour market "tightness", i.e., pressure from the proletariat (the working class), and supply and demand. Labour market tightness is used by Goodwin in his growth cycle as the primary cause of fluctuations in capital. He defines its derivative as:

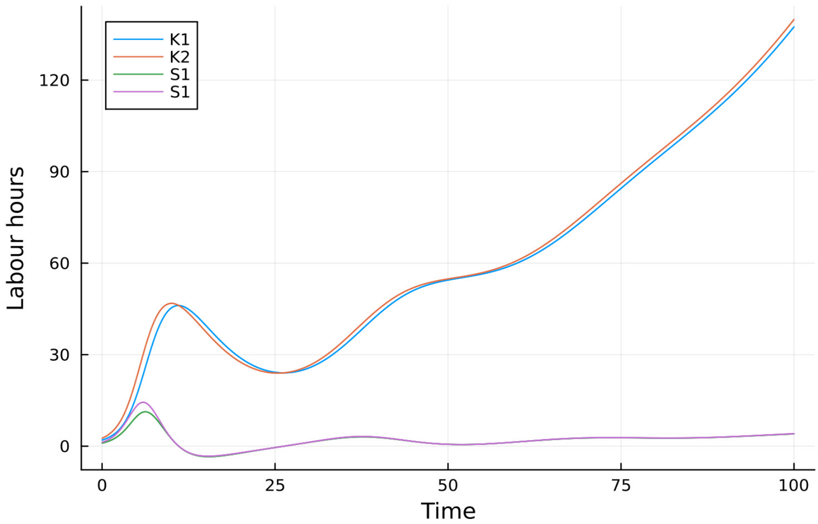
$$\frac{dW}{dt} = W(t)(\rho v - \gamma) \tag{15}$$

in which:

ρ, γ – positive constants

v – employment rate.

In this section, we will simulate an economy by using this definition of wage (see Appendix C for the code). All other parameters will be the same as initially defined in our first numerical solution to the system. Our wage will have an initial value of 1. A new parameter will also be introduced in the model: population, p, which will have a constant growth rate defined by the coefficient β. The span of work time periods will be longer, $t \in [0,100]$. The result can be seen in Figure 5.



Source: Author's calculation.

Figure 5. Goodwin's definition of wage

$$p_0 = 100, \beta = 0.02, \rho = 1, \gamma = 0.1$$

As discussed in the literature review, Goodwin's model states that initial fluctuations in capital and output will occur, but those fluctuations should slowly converge to a stable growth state, which we can also observe in Figure 5, making our model consistent with Goodwin's theory. I'm merely noting this as a characteristic of my model, rather than stating that my objective was to create a model consistent with Goodwin's.

The cause of the fluctuations observed above is the fluctuating wage, which decreases the surplus value, sometimes till the point of a negative value, which in turn decreases the change in capital stock. The opposite happens afterwards. This cycle repeats itself until a stable growth state is reached.

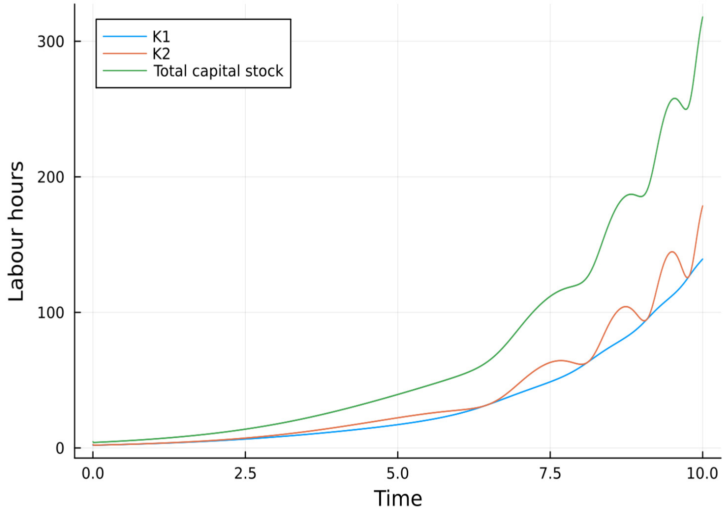
Introduction of Non-Constant, Unproductive Consumption in Sector I

We shall now define our parameters as the ones previously stated in the first numerical example, except for a_1, which I will define as a fluctuating (between 0 and 1) function. I will assume a_1 to be dependent on the current capital stock of Sector I in such a way that:

$$a_1(t) = \alpha \sin^2(\beta\, K(t - \Delta t)) \tag{16}$$

wherein the sine function is squared; otherwise, a_1 would fluctuate between -1 and 1, which is not desirable. In the following simulation (Figure 6), I have set:

$\alpha = 0.3$ and $\beta = 0.1$.



Source: Author's calculation.

Figure 6. Introduction of a fluctuating value of a_1

The above pattern is a cyclical one, which was expected. Fluctuations in the capital stock of Sector I are directly caused by fluctuations in a_1, which cause the derivative of K_1 to shift between 0 and 1; S_1 never reaches a negative value, though. On the other hand, the capital stock of Sector II is reacting in the completely opposite direction. Increases in a_1 lead to the capitalists from Sector I purchasing a greater amount of consumer goods, increasing the capital stock in that sector. The opposite happens when the value is reduced.

Of course, one might say that even if fluctuations in a_1 cause these cyclical patterns in one sector, the expenditures of one capitalist are the income of another: therefore, the two capital stocks combined do not undergo any cyclical patterns, and if one sector lays off a massive amount of workers, the other sector would hire them. That is why I have plotted the sum of the two capital stocks in Figure 6. As can be seen, the latter argument is disproved. The capital stock of the entire economy is still undergoing crises in a cyclical pattern, causing massive unemployment.

Conclusion

We have succeeded in the construction of a macroeconomic model of capital accumulation in an economy with two sectors in this paper. The model developed has properties that can be found in the expanded versions of Goodwin's model, and by that I am referring to the fact that my model accounts for technical changes, as well as properties which are a characteristic of Foley's model – the introduction of two sectors in the economy.

While the system can be solved analytically, it is preferrable to solve it numerically as this gives the user the possibility of setting the parameters – such as wages, the organic composition of capital and so on – according to either empirical data or some other relationship that cannot be perfectly described by an analytically defined function.

After performing a sensitivity analysis on the parameters which were initially assumed to be constant, it was discovered, according to this model, that an economy (or just one sector) can undergo crises with different causes. These are:

- technical changes in Sector I
- technical changes in Sector II
- upper pressure from the working class on wages
- fluctuations in the portion of surplus being unproductively consumed in Sector I.

The model, of course, has its weaknesses, one of which is the fact that it doesn't consider the different forms of capital – whether money, productive capital or commodity capital. The model also abstracts itself from the effects of changes in the value of money, the application of fiscal policies and the government's role as a whole. This model could be expanded in order to account for other parameters.

I believe this model can be used by other economists in the future for the purpose of validating hypotheses which assume some concretely defined parameters, like the wages in Goodwin's model.

Conflicts of Interest

The author has no conflicts of interest to declare.

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Appendix A: Code for initial numerical simulation

```
using Plots

T = 10 #length of the w-p dimension
n = 1000 #number of steps
dt = T/n #delta t
t = range(0,T,n) #w-p dimension array

a = 0.3
c1 = ones(n)
c2 = ones(n)
W = ones(n)
L = 2
δ = 0.8

#I Sector
Q1 = zeros(n)
K1 = zeros(n)
V1 = zeros(n)
S1 = zeros(n)

V1[1] = 1
K1[1] = V1[1]*(1+c1[1])
Q1[1] = δ*c1[1]*V1[1] + (V1[1]*L)/W[1]
S1[1] = (V1[1]*L)/W[1] - V1[1]
#II Sector
Q2 = zeros(n)
K2 = zeros(n)
V2 = zeros(n)
S2 = zeros(n)
```

```
V2[1] = (V1[1]+a*S1[1])*(1/c2[1])
K2[1] = (V1[1]+a*S1[1])*(1 + 1/c2[1])
Q2[1] = V2[1]*c2[1]*δ + (V2[1]/W[1])*L
S2[1] = (V2[1]/W[1])*L - V2[1]

for i in 2:n
    #Sector I
    K1[i] = K1[i-1] + V1[i-1]*(L/W[i] - 1)*(1-a)*dt
    V1[i] = K1[i]/(1+c1[i])
    Q1[i] = δ*c1[i]*V1[i] + (V1[i]*L)/W[i]
    S1[i] = (V1[i]*L)/W[i] - V1[i]
    #Sector II
    K2[i] = (V1[i]+a*S1[i])*(1 + 1/c2[i])
    V2[i] = (V1[i]+a*S1[i])*(1/c2[i])
    Q2[i] = V2[i]*c2[i]*δ + (V2[i]/W[i])*L
    S2[i] = (V2[i]/W[i])*L - V2[i]
end

plot(t,K1, label="Capital Stock of Sector I")
plot!(t,K2, label="Capital Stock of Sector II")
xlabel!("Time")
ylabel!("Labour hours")
```

Appendix B: Introduction of technical changes in the code

For the technical change in Sector I, the code was only changed in one place by adding

```
c1[500:1000] .= 4
```

after c_1 had been defined.

For the technical change in Sector II, the code was only changed in one place by adding

```
c2[500:1000] .= 4
```

after c_2 had been defined.

For the technical changes in both sectors, the code is nothing but a combination of the two above changes.

```
c1[500:1000] .= 4
c2[500:1000] .= 4
```

Appendix C: Introduction of a non-constant wage in the code

```
T = 100  #length of the w-p dimension
n = 1000 #number of steps
dt = T/n #delta t
t = range(0,T,n) #w-p dimension array

a = 0.3
c1 = ones(n)
c2 = ones(n)
W = ones(n)
L = 2
δ = 0.8
ρ = 1
γ = 0.1
p = 100*ones(n) #population
β = 0.02

#I Sector
Q1 = zeros(n)
K1 = zeros(n)
V1 = zeros(n)
S1 = zeros(n)

V1[1] = 1
K1[1] = V1[1]*(1+c1[1])
Q1[1] = δ*c1[1]*V1[1] + (V1[1]*L)/W[1]
S1[1] = (V1[1]*L)/W[1] - V1[1]
##

#II Sector
Q2 = zeros(n)
K2 = zeros(n)
V2 = zeros(n)
S2 = zeros(n)
```

```
V2[1] = (V1[1]+a*S1[1])*(1/c2[1])
K2[1] = (V1[1]+a*S1[1])*(1 + 1/c2[1])
Q2[1] = V2[1]*c2[1]*δ + (V2[1]/W[1])*L
S2[1] = (V2[1]/W[1])*L - V2[1]

for i in 2:n
    p[i] = p[i-1] + p[i-1]*β*dt #change in population
    W[i] = W[i-1] + W[i-1]*(ρ*((V1[i-1]+V2[i-1])/W[i-1])/p[i-1] -
γ)*dt #Goodwin's wage definition
    #Sector I
    K1[i] = K1[i-1] + V1[i-1]*(L/W[i] - 1)*(1-a)*dt
    V1[i] = K1[i]/(1+c1[i])
    Q1[i] = δ*c1[i]*V1[i] + (V1[i]*L)/W[i]
    S1[i] = (V1[i]*L)/W[i] - V1[i]
    #Sector II
    K2[i] = (V1[i]+a*S1[i])*(1 + 1/c2[i])
    V2[i] = (V1[i]+a*S1[i])*(1/c2[i])
    Q2[i] = V2[i]*c2[i]*δ + (V2[i]/W[i])*L
    S2[i] = (V2[i]/W[i])*L - V2[i]
end
```