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# HEURISTIC APPROACH FOR DETERMINING EFFICIENT FRONTIER PORTFOLIOS WITH MORE THAN TWO ASSETS, THE CASE OF ZSE

The goal of this paper is to exhibit computation of minimal variance portfolio and efficient portfolio frontier when there are more than two assets, by using matrix algebra applied on chosen stocks listed on Zagreb Stock Exchange. The research shows that, because of low correlation of underlying assets, it is possible to significantly reduce risks of investments by constructing portfolio of the stocks. It also shows that, if restriction on short selling are imposed this significantly reduces the possibility for diversification.

JEL: G11; G31; C02

In literature (Ross, 1976; Chen, Roll, Ross, 1986; Van Horne, Wachowicz, 2008) efficient portfolio computations are usually shown based on two assets while situation with more than two assets are presented more generally without detailed computations. If we want to use the modern portfolio theory (MPT) in the real world, than we have to deal with the case where there are more than two assets. The goal of this paper is to show how to determine efficient portfolio when we have more than two assets by using matrix algebra applied on chosen stocks listed on Zagreb Stock Exchange. Although in practice special software is usually used for efficient portfolio calculation, the knowledge of mathematical procedures still can be very useful for understanding the theoretical logic of modern portfolio theory.

MPT is a mathematical formulation of the concept of diversification in investing, with the aim of selecting a collection of investment assets that has collectively lower risk than any individual asset. If investor has to choose between two portfolios with the same expected return, he will prefer the less risky one. The portfolio with higher risk will be chosen only if it offers adequately higher expected returns, and investor who wants higher expected returns must accept more risk. The exact trade-off will be evaluated differently, based on investors' risk aversion characteristics. So, MPT assumes that a rational investor will not invest in one portfolio if a second portfolio exists with a more favourable risk-expected return profile or, if for a given level of risk an alternative portfolio exists that has better expected returns (Chen et al., 1990). The theory uses standard deviation of return as a proxy for risk, which is valid if assets returns are jointly normally distributed or otherwise elliptically distributed. Portfolio return is the proportion-weighted combination of the constituent assets' returns. Portfolio volatility is a function of the correlations ($\rho_{ij}$) of the component assets, for all given asset pairs. An efficient portfolio is the portfolio of assets that gives the lowest variance of return of all portfolios having the same expected return. Alternatively, it can be said that an

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efficient portfolio has the highest expected return of all portfolios having the same variance (Benninga, 2000).

Our example is based on a problem of estimating the minimum variance portfolio and the efficient frontier portfolios, which comprises of selected shares from the Zagreb Stock Exchange. The selection criteria are: 1) the share has to be listed at the Official Market of ZSE; 2) it has to be part of CROBEX10 stock market index; and 3) it should have positive returns in average for the observed period. Analysed period is May 2009 to May 2014.

## Literature review

Modern portfolio theory and portfolio selection has got a lot of attention in both academic and practitioner literatures since seminal work of Markowitz (1952). He provides a normative framework through which optimal portfolios may be identified. Optimal portfolio selection requires knowledge of each security's expected return, variance, and covariance with other security returns. In practice, these data are unknown and must be estimated from available historical or subjective information. In that case estimation error can degrade the desirable properties of the investment portfolio that is selected. This problem of estimation risk in portfolio selection was investigated by Klein et.al. (1976). They showed that for normally distributed returns and ‘non-informative’ or ‘invariant’ priors, the admissible set of portfolios taking the estimation uncertainty into account was identical to that given by traditional analysis. However, as a result of estimation risk, the optimal portfolio choice differs from that obtained by traditional analysis. For other plausible priors, the admissible set, and consequently the optimal choice, was shown to differ from that in traditional analysis.

Tobin (1958) showed that under certain conditions Mrakowitz’s model implies that the process of investment choice can be broken down into two phases: first, the choice of a unique optimum combination of risky assets; and second, a separate choice concerning the allocation of funds between such a combination and a single riskless asset.

Sharpe (1963) describes the advantages of using a diagonal model of the relationships among securities for practical applications of the Markowitz portfolio analysis technique. He developed a computer program in order to take full advantage of the model. Based on the research results he concluded that diagonal model may be able to represent the relationships among securities rather well and that the value of portfolio analyses based on the model will exceed their nominal cost.

Gennotte (1986) investigated optimal portfolio choice under incomplete information. In his paper he derived optimal estimators for the unobservable expected instantaneous returns using observations of past realized returns, and showed how to solve portfolio choice in two separate steps. He also analysed the impact of estimation error on investment choices. He concluded that the effects of the estimation error should not be ignored even if realized returns are observed continuously over an infinite time period.

Davis and Norman (1990) studied portfolio selection with transaction costs. They concluded that optimal buying and selling investment selection between bank account paying a fixed rate of interest and a stock whose price is a log-normal diffusion, are the local times of the two-dimensional process of a wedge-shaped region which is determined by the solution of a nonlinear free boundary problem. They also offered the algorithm for solving that problem.

Many more papers have been trying different mathematical approaches to develop the theory of portfolio model (Zimmermann et.al, 2001; Hung et.al, 1996). In particularly, the performance measure of portfolio has become one of the most important factors in theoretical and practical investment. As recent studies in the sense of mathematical programming, some researchers have proposed various types of portfolio models under randomness and fuzziness (Lin and Hsieh, 2004; Parra et.al., 2001).

## Model specification

According to the initial selection criteria the investor has to make the decision about the investments in four stocks of ZSE: AD Plastik (ADPL), Atlantic Grupa dd (ATGR), Ledo dd (LEDO) and Podravka dd (PODR). $R_i$ (i=ADPL, ATGR, LEDO, PODR) denote the monthly continuously compound (CC) return on stock *i*. We assume that $R_i$ are jointly normally distributed $R_i \sim N(\mu_i, \sigma_i^2)$, and that we have the following information about the means, variances and covariances of the probability distribution of the returns:

(1) $\mu_i = E[R_i]$

(2) $\sigma_i^2 = Var(R_i)$

(3) $cov(R_i, R_j) = \sigma_{ij}$

Value of $\mu_i$ and $\sigma_i^2$ can be estimated from historical return data. Since the returns are random the realized returns may be different from our expectations, so the variances provide the measures of the uncertainty (or risk) associated with these monthly returns (Zivot, 2002).

Portfolio *X* consists of four assets whose portfolio weights add up to 1:

(4) $x_{ADPL} + x_{ATGR} + x_{LEDO} + x_{PODR} = 1$

Portfolio CC return is a weighted average of the returns on the four different assets:

(5) $R_{p,x} = x_{ADPL}R_{ADPL} + x_{ATGR}R_{ATGR} + x_{LEDO}R_{LEDO} + x_{PODR}R_{PODR}$

In matrix algebra we have:

(6) $$R = \begin{bmatrix} R_{ADPL} \\ R_{ATGR} \\ R_{LEDO} \\ R_{PODR} \end{bmatrix}, \mu = \begin{bmatrix} \mu_{ADPL} \\ \mu_{ATGR} \\ \mu_{LEDO} \\ \mu_{PODR} \end{bmatrix}, 1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, x = \begin{bmatrix} x_{ADPL} \\ x_{ATGR} \\ x_{LEDO} \\ x_{PODR} \end{bmatrix}$$

(7) $$\Sigma = \begin{bmatrix} \sigma_{ADPL}^2 & \sigma_{ADPL,ATGR} & \sigma_{ADPL,LEDO} & \sigma_{ADPL,PODR} \\ \sigma_{ATGR,ADPL} & \sigma_{ATGR}^2 & \sigma_{ATGR,LEDO} & \sigma_{ATGR,PODR} \\ \sigma_{LEDO,ADPL} & \sigma_{LEDO,ATGR} & \sigma_{LEDO}^2 & \sigma_{LEDO,PODR} \\ \sigma_{PODR,ADPL} & \sigma_{PODR,ATGR} & \sigma_{PODR,LEDO} & \sigma_{PODR}^2 \end{bmatrix}$$

Vector R is the return vector, vector μ is expected return vector, 1 represents a vector of ones, x is a portfolio vector, Σ is covariance matrix which summarizes the variances and the covariances. The diagonals are the squares of the volatilities, and the off-diagonals are the covariances.

(8) $$x'1 = \begin{bmatrix} x_{ADPL} & x_{ATGR} & x_{LEDO} & x_{PODR} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1$$

Portfolio return can be written as x'R:

(9) $$R_{p,x} = x'R = \begin{bmatrix} x_{ADPL} & x_{ATGR} & x_{LEDO} & x_{PODR} \end{bmatrix} \begin{bmatrix} R_{ADPL} \\ R_{ATGR} \\ R_{LEDO} \\ R_{PODR} \end{bmatrix}$$

Similarly, portfolio expected return is x'μ:

(10) $$\mu_{p,x} = x'\mu = \begin{bmatrix} x_{ADPL} & x_{ATGR} & x_{LEDO} & x_{PODR} \end{bmatrix} \begin{bmatrix} \mu_{ADPL} \\ \mu_{ATGR} \\ \mu_{LEDO} \\ \mu_{PODR} \end{bmatrix}$$

Portfolio variance in matrix notation is:

(11) $$\sigma_{p,x}^2 = x'\Sigma x$$

$$\sigma_{p,x}^2 = \begin{bmatrix} x_{ADPL} & x_{ATGR} & x_{LEDO} & x_{PODR} \end{bmatrix} \begin{bmatrix} \sigma_{ADPL}^2 & \sigma_{ADPL,ATGR} & \sigma_{ADPL,LEDO} & \sigma_{ADPL,PODR} \\ \sigma_{ATGR,ADPL} & \sigma_{ATGR}^2 & \sigma_{ATGR,LEDO} & \sigma_{ATGR,PODR} \\ \sigma_{LEDO,ADPL} & \sigma_{LEDO,ATGR} & \sigma_{LEDO}^2 & \sigma_{LEDO,PODR} \\ \sigma_{PODR,ADPL} & \sigma_{PODR,ATGR} & \sigma_{PODR,LEDO} & \sigma_{PODR}^2 \end{bmatrix} \begin{bmatrix} x_{ADPL} \\ x_{ATGR} \\ x_{LEDO} \\ x_{PODR} \end{bmatrix}$$

Portfolio distribution is:

(12) $$R_{p,x} \sim N(\mu_{p,x}, \sigma_{p,x}^2)$$

In order to determine efficient frontier we need to calculate covariance between two portfolio returns. The portfolios differ only in the weights, but not in stock included in portfolios. The portfolio weights in both portfolios add to 100%

Let *y* denote another portfolio of underlying stocks:

(13) $$y = \begin{bmatrix} y_{ADPL} \\ y_{ATGR} \\ y_{LEDO} \\ y_{PODR} \end{bmatrix}$$

Then portfolio y has returns $R_{p,y}$,

(14) $$R_{p,y} = y'R$$

So, if we have two portfolios, where one portfolio is *x*, another is *y*, the covariance between these two portfolios is:

(15) $cov(R_{p,x}, R_{p,y}) = x'\Sigma y = y'\Sigma x$

In order to calculate the correlation between portfolios we can use (16):

(16) $corr(R_{p,x}, R_{p,y}) = \frac{cov(R_{p,x}, R_{p,y})}{\sigma_{p,x} \cdot \sigma_{p,y}}$

Global minimum variance portfolio is located on the efficient frontier of risky assets and it has the smallest variance. In order to find the analytic solution for it when we have more than two assets the best way is to employ the matrix algebra.

So, in this example we have to calculate the weights of $m = [m_{ADPL} \quad m_{ATGR} \quad m_{LEDO} \quad m_{PODR}]'$ portfolio that solves:

(17) $\min_{m_{ADPL},\ m_{ATGR},\ m_{LEDO},\ m_{PODR}} \sigma_{p,m}^2 = m'\Sigma m$

With constraint that portfolio weights sum is 1:

(18) $m'1 = 1$

To do that we have to set up the LaGrangian function (Bertsekas 1999; Chiang, A. C., 1984) The LaGrangian function for this problem is:

(19) $L(m, \lambda) = m'\Sigma m + \lambda(m'1 - 1)$

We put constrain in homogenous form, we multiply it by LaGrange multiplier and then we add the term to be minimized. The LaGrangian function is a function of portfolio weights and the LaGranger multiplier lambda. We derive the LaGrangian in with respect to m and λ, and set it equal to zero:

(20) $0 = \frac{\partial L(m,\lambda)}{\partial m} = \frac{\partial m'\Sigma m}{\partial m} + \frac{\partial}{\partial m}\lambda(m'1 - 1) = 2\Sigma m + \lambda 1$

(21) $0 = \frac{\partial L(m,\lambda)}{\partial \lambda} = \frac{\partial m'\Sigma m}{\partial \lambda} + \frac{\partial}{\partial \lambda}\lambda(m'1 - 1) = m'1 - 1$

The first order conditions give us two linear equations. We can merge them and write them as a system of linear equations:

(22) $\begin{bmatrix} 2\Sigma & 1 \\ 1' & 0 \end{bmatrix} \begin{bmatrix} m \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The first order conditions are a linear system:

(23) $A_m z_m = b$, were $A_m = \begin{bmatrix} 2\Sigma & 1 \\ 1' & 0 \end{bmatrix}$, $z_m = \begin{bmatrix} m \\ \lambda \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The solution for $z_m$ is:

(24) $z_m = A_m^{-1} b$

The first four elements of $z_m$ are portfolio weights $m = [m_{ADPL} \quad m_{ATGR} \quad m_{LEDO} \quad m_{PODR}]'$ for the global minimum variance portfolio with expected return $\mu_{p,m} = m'\mu$ and variance $\sigma_{p,m}^2 = m'\Sigma m$.

## Data

Our example is based on problem of estimating the Minimum variance portfolio and the efficient frontier portfolios, which comprises the selected shares listed at the Official Market included in CROBEX10 index of Zagreb Stock Exchange. According to the mentioned initial set of criteria (see Introduction) the following shares were selected: AD Plastik (ADPL), Atlantic Grupa dd (ATGR), Ledo dd (LEDO) and Podravka dd (PODR). With regard to the movement of stock prices the continuously compounded (CC) monthly returns were calculated. Time series comprises 60 monthly CC returns. The data source for calculation is Zagreb Stock Exchange. Descriptive statistics for selected stocks is given in Table 1.

*Table 1*

Descriptive statistics

| | ADPL | ATGR | LEDO | PODR |
|---|---|---|---|---|
| Mean | 0.011510 | 0.008867 | 0.011212 | 0.011969 |
| Median | 0.014004 | 0.010233 | 0.002496 | 0.005899 |
| Maximum | 0.170886 | 0.183251 | 0.166665 | 0.173440 |
| Minimum | -0.073094 | -0.110791 | -0.118130 | -0.126023 |
| Std. Dev. | 0.059560 | 0.054831 | 0.055979 | 0.066849 |
| Skewness | 0.557660 | 0.445303 | 0.345902 | 0.257397 |
| Kurtosis | 2.665580 | 3.932845 | 3.254464 | 2.795047 |
| Jarque-Bera | 3.389443 | 4.158449 | 1.358362 | 0.767547 |
| Probability | 0.183650 | 0.125027 | 0.507032 | 0.681286 |
| Sum | 0.690620 | 0.532027 | 0.672711 | 0.718164 |
| Sum Sq. Dev. | 0.209299 | 0.177380 | 0.184885 | 0.263658 |
| Observations | 60 | 60 | 60 | 60 |

*Source*. Author's calculation using software package EViews 7.0.

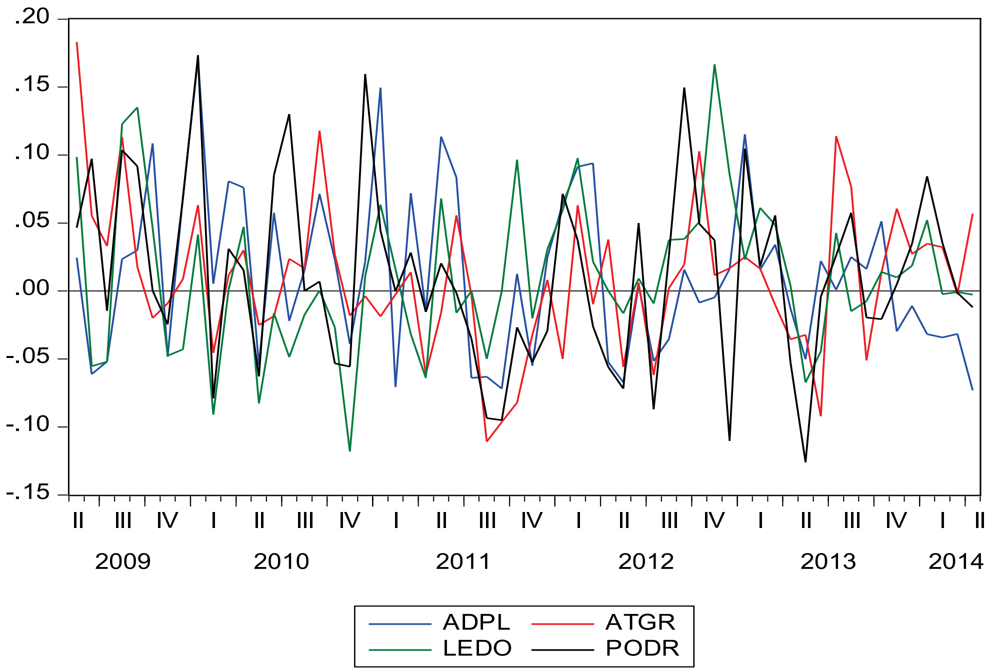
From Table 1 it can be noticed that the all distributions have a long right tail (positive skewness). In addition, ADPL and PODR distributions are flat (platykurtic) relative to the normal distribution, while distributions of ATGR and LEDO is peaked (leptokurtic) relative to the normal. However, the reported probability of Jarque-Bera test for all variables is above 0,05 and that means that we cannot reject the null hypothesis of Jarque-Bera test that variable is normal distributed at the 5% level significance[1]. The monthly average CC returns of stocks during analysed period were between 0.8% and 1.2% (all positive). If we annualize this that means that the annual returns were between 9.6% and 14.4%. Series had monthly standard deviations of around 6%.

Figure 1 contains Monthly CC returns of chosen assets. This time series looks like Gaussian white noise. The volatility of all series seems constant (covariance stationary). The data appears to fluctuate close to zero without any noticeable trend. It seems that there are positive correlations between assets.

[1] In another words, this means that we should accept the null hypothesis of Jarque-Berra test that variable is normal distributed at the 5% significance.

Figure 1

Monthly CC returns of AD Plastik (ADPL), Atlantic Grupa dd (ATGR), Ledo dd (LEDO) and Podravka dd (PODR)




*Source*. Author's calculation using software package EViews 7.0.

Further analysis depends on assumption of weak stationarity of the series. Weak stationarity of time series means that the time series has a constant mean, a constant variance, and that the covariance between two time periods depends only on the interval, and not the timing. The importance of stationarity of time series is reflected in the possibility of generalisation of dynamic sample analysis to a whole population. A test that is used for testing stationarity is Unit Root Test developed by Dickey and Fuller (1979).

The results of the tests are presented in Table 2. Null hypothesis is set: $H_0$ – series is nonstationary (has a unit root). This hypothesis cannot be rejected if the calculated ADF value is greater than the critical value. As shown in the table, the ADF test suggest for all variables to reject the $H_0$ hypothesis with a significance level of 1%. In order to avoid autocorrelation of residuals in autoregressive model, lag was selected based upon Akaike information criterion (AIC) and also is given in *Table 2*.

*Table 2*

Unit root test

| Model: | Intercept | | Trend and intercept | | None | |
|---|---|---|---|---|---|---|
| ADF test | ADF | Lag | ADF | Lag | ADF | Lag |
| ADPL | -6.828415*** | 0 | -7.1108*** | 0 | -6.6776*** | 0 |
| ATGR | -6.726417*** | 0 | -6.5931*** | 0 | -6.7361*** | 0 |
| LEDO | -6.159686*** | 0 | -6.2444*** | 0 | -6.0836*** | 0 |
| PODR | -6.846060*** | 0 | -6.9342*** | 0 | -6.7350*** | 0 |
| | 1% test critical level: -3.5461,<br>5% test critical level: -2.9117,<br>10% test critical level: -2.5935 | | 1% test critical level: -4.1213,<br>5% test critical level: -3.4878,<br>10% test critical level: -3.1723 | | 1% test critical level: -2.6047,<br>5% test critical level: -1.9464,<br>10% test critical level: -1.6132 | |

*Note:* *** - rejection of H0 at 1% of Critical level; ** - rejection of H0 at 5% of Critical level; * - rejection of H0 at 10% of Critical level.

The other Unit root tests (PP, KPSS) also were carried out. The results of these tests are not presented here. However, they also suggest the same conclusions.

*Source.* Author's calculation using software package EViews 7.0.

Covariance measures the direction but not the strength of linear association. We can see from Table 3 that all variables have positive covariance, and that means that they tend to move in the same direction.

*Table 3*

Covariance and correlation analysis

| Covariance Analysis: Ordinary | | Sample: 2009M05 2014M04 | | Observ.: 60 |
|---|---|---|---|---|
| Covariance | ADPL | ATGR | LEDO | PODR |
| ADPL | 0.003488 | | | |
| ATGR | 0.000642 | 0.002956 | | |
| LEDO | 0.001206 | 0.001004 | 0.003081 | |
| PODR | 0.001744 | 0.001625 | 0.001329 | 0.004394 |
| Correlation | ADPL | ATGR | LEDO | PODR |
| ADPL | 1.000000 | | | |
| ATGR | 0.199959 | 1.000000 | | |
| LEDO | 0.367852 | 0.332778 | 1.000000 | |
| PODR | 0.445480 | 0.450760 | 0.361249 | 1.000000 |

*Source.* Author's calculation using software package EViews 7.0.

If we want to know not only the direction but also the strength of relationship we need to calculate correlation. By the definition of Pearson correlation, it is just the covariance divided by the square root of the product of the variances. The Pearson correlation is +1 in the case of a perfect direct (increasing) linear relationship (correlation), −1 in the case of a perfect decreasing (inverse) linear relationship (anticorrelation), and some value between −1 and +1 in all other cases, indicating the degree of linear dependence between the variables. As it approaches zero there is less of a relationship (closer to uncorrelated). The closer the coefficient is to either −1 or +1, the stronger the correlation is between the variables (Dowdy, Wearden, 1983).

From Table 3 it can be noticed that there is no strong correlation between any underlying assets and this is a good property for portfolio diversification. The data from Table 3 will be used further in calculation of portfolio variance and covariance.

## Empirical results

*Return-risk characteristics of random portfolios*

Based on calculation of CC return for ADPL, ATGR, LEDO and PODR (see Figure 1), we calculated expected return μ and covariance matrix Σ (see Table 1 and 3).

$$\mu = \begin{bmatrix} 0.01151 \\ 0.008867 \\ 0.011212 \\ 0.011969 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.00349 & 0.00064 & 0.00121 & 0.00174 \\ 0.00064 & 0.00297 & 0.00100 & 0.00163 \\ 0.00121 & 0.00100 & 0.00308 & 0.00133 \\ 0.00174 & 0.00163 & 0.00133 & 0.00440 \end{bmatrix}$$

We can calculate variance, standard deviation and expected return for equally weighted portfolio ($x_{EW}$) and arbitrary chosen long-short portfolio ($y_{LS}$) using (10) and (11), and then we can calculate the covariance and correlation between these two portfolios using (15) and (16). The long-short portfolio means that portfolio weights had to add up to one, but some can be negative, and that means that investor shorting the asset, and taking the proceeds of that short sale to buy more of other assets.

The chosen portfolios are:

$$x_{EW} = \begin{bmatrix} x_{ADPL} \\ x_{ATGR} \\ x_{LEDO} \\ x_{PODR} \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix} \quad y_{LS} = \begin{bmatrix} y_{ADPL} \\ y_{ATGR} \\ y_{LEDO} \\ y_{PODR} \end{bmatrix} = \begin{bmatrix} -0.09 \\ -0.28 \\ 0.67 \\ 0.70 \end{bmatrix}$$

The results:

$$R_{p,x_{EW}} x'_{EW} \cdot R = 0.0109,$$

$$R_{p,y_{LS}} = y'_{LS} \cdot R = 0.0124,$$

$$\sigma^2_{p,x_{EW}} = x'_{EW} \cdot \Sigma \cdot x_{EW} = 0.00181, \ \sigma_{p,x_{EW}} = 0.0425,$$

$$\sigma^2_{p,y_{LS}} = y'_{LS} \cdot \Sigma \cdot y_{LS} = 0.00374, \sigma_{p,y_{LS}} = 0.0612,$$

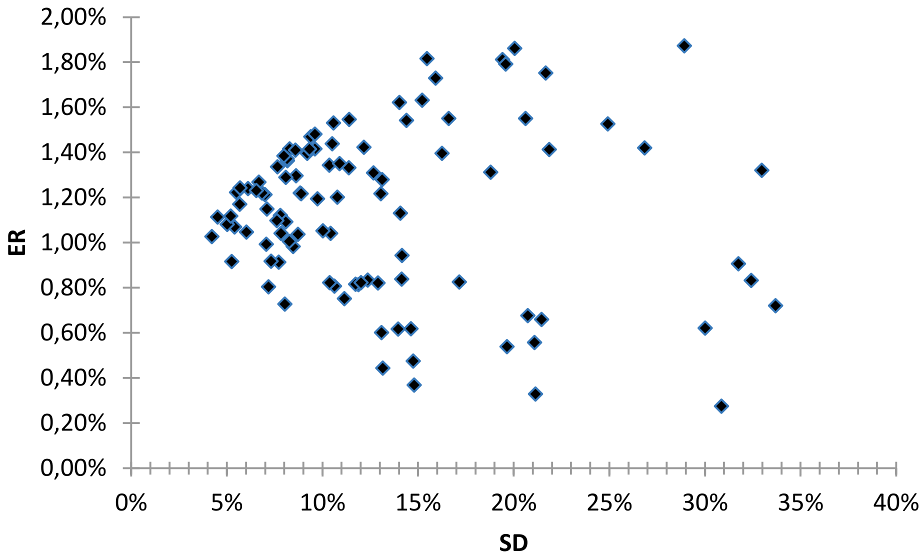
$$cov\left(R_{p,x_{EW}}, R_{p,y_{LS}}\right) = x'_{EW} \cdot \Sigma \cdot y_{LS} = y'_{LS} \cdot \Sigma \cdot x_{EW} = 0.0021$$

$$corr\left(R_{p,x_{EW}}, R_{p,y_{LS}}\right) = \frac{cov\left(R_{p,x_{EW}}, R_{p,y_{LS}}\right)}{\sigma_{p,x_{EW}} \cdot \sigma_{p,y_{LS}}} = 0.808$$

We can see that equally weighted portfolio and long-short portfolio have positive covariance, and the correlation between them is 0.808, and that means that they are highly positively correlated.

Figure 2

Return-risk characteristics of random portfolios




*Source.* Author's calculation using software package EViews 7.0.

Figure 2 shows the return-risk trade-offs. X axis measures volatility (level of risk), and Y axis measures return. Based on four assets (ADPL, ATGR, LEDO and PODR) we have generated 100 random portfolios, similarly as $x_{EW}$ and $y_{LS}$, which weights add up to 1. Each portfolio is presented with one dot on Figure 2. We can think of this as possible portfolios that one could invest in. We can notice that there are clusters of portfolios with the same expected returns, but different standard deviations, and also a lot of portfolios with the same standard deviations and different expected returns. It is much better to invest in portfolios which are located in the upper left corner then in the lower right corner, because the former have a higher average return and the lower risk.

Figure 2 shows that when we move away from the two asset case, and when we have three or more assets, the risk-return trade-offs between portfolios no longer lies on a straight line, but in fact fills an entire space. If we calculate these portfolios more times than this, then we would get a half-moon shaped kind of form that would contain the universe of potential portfolios that we could invest in. However, the upper boundary on that shape is efficient portfolios. Markowitz (1991) provided the mathematics for determining the boundary of this investment possibility shape that determines the set of the efficient portfolios.

*Global minimum variance portfolio*

Now, we can use (23) and (24) to calculate

$m = [m_{ADPL} \quad m_{ATGR} \quad m_{LEDO} \quad m_{PODR}]'$:

$$A_m = \begin{bmatrix} 0.00698 & 0.00128 & 0.00241 & 0.00349 & 1 \\ 0.00128 & 0.00591 & 0.00200 & 0.00325 & 1 \\ 0.00241 & 0.00200 & 0.00616 & 0.00266 & 1 \\ 0.00349 & 0.00349 & 0.00266 & 0.00879 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \ z_m = \begin{bmatrix} m_{ADPL} \\ m_{ATGR} \\ m_{LEDO} \\ m_{PODR} \\ \lambda \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$z_m = A_m^{-1} b = \begin{bmatrix} 165.753 & -22.306 & -75.637 & -67.810 & 0.291 \\ -22.306 & 177.828 & -78.523 & -76.999 & 0.385 \\ -75.637 & -78.523 & 183.318 & -29.159 & 0.288 \\ -67.810 & -76.999 & -29.159 & 173.968 & 0.035 \\ 0.291 & 0.385 & 0.288 & 0.035 & 0.003 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.291 \\ 0.385 \\ 0.288 \\ 0.035 \\ -0.003 \end{bmatrix}$$

So, the minimal variance portfolio has 29.1% AD Plastik, 38.5% Atlantic Grupa dd, 28.8% Ledo and 3.5% Podravka dd.

The expected return, variance and standard deviation on this portfolio are:

$$\mu_{p,m} = m'\mu = [0.291 \quad 0.385 \quad 0.288 \quad 0.035] \begin{bmatrix} 0.01151 \\ 0.008867 \\ 0.011212 \\ 0.011969 \end{bmatrix} = 0.01042$$

$$\sigma_{p,m}^2 = m'\Sigma m =$$

$$[0.291 \quad 0.385 \quad 0.288 \quad 0.035] \begin{bmatrix} 0.00349 & 0.00064 & 0.00121 & 0.00174 \\ 0.00064 & 0.00297 & 0.00100 & 0.00163 \\ 0.00121 & 0.00100 & 0.00308 & 0.00133 \\ 0.00174 & 0.00163 & 0.00133 & 0.00440 \end{bmatrix} \begin{bmatrix} 0.291 \\ 0.385 \\ 0.288 \\ 0.035 \end{bmatrix} = 0.00168$$

$$\sigma_{p,m} = 0.0409$$

From (23) and (24) we can see that Minimum variance portfolio only depends upon the covariance matrix, and not upon the expected returns.

*Portfolio with target expected return and with the smallest variance*

Another problem which appears in practice is to determine portfolio that has the smallest risk, measured by portfolio variance that achieves a target expected return. This problem is actually only the expansion of problem (17), with additional constraint:

(25) $\min_{x_{ADPL},\ x_{ATGR},\ x_{LEDO},\ x_{PODR}} \sigma_{p,x}^2 = x'\Sigma x$

(26) $\mu_{p,x} = x'\mu = \mu_p^0 = target\ return$

(27) $x'1 = 1$

As same as before, we can set up the LaGrangian function for this problem:

$$L(x, \lambda_1, \lambda_2) = x'\Sigma x + \lambda_1(x'\mu - \mu_{p,0}) + \lambda_2(x'1 - 1)$$

(28) $$0 = \frac{\partial L(x,\lambda_1,\lambda_2)}{\partial x} = 2\Sigma x + \lambda_1 \mu + \lambda_2 1$$

(29) $$0 = \frac{\partial L(x,\lambda_1,\lambda_2)}{\partial \lambda_1} = x'\mu - \mu_{p,0}$$

(30) $$0 = \frac{\partial L(x,\lambda_1,\lambda_2)}{\partial x} = x'1 - 1$$

We can represent the first order conditions in matrix notation as linear system:

(31) $$A_x z_x = b_0, \text{ were } A_x = \begin{bmatrix} 2\Sigma & \mu & 1 \\ \mu' & 0 & 0 \\ 1' & 0 & 0 \end{bmatrix},\ z_x = \begin{bmatrix} x \\ \lambda_1 \\ \lambda_2 \end{bmatrix},\ b_0 = \begin{bmatrix} 0 \\ \mu_{p,0} \\ 1 \end{bmatrix}$$

The solution for $z_x$ is:

(32) $$z_x = A_x^{-1} b_0$$

The first four elements of $z_x$ give us optimal portfolio weights $x = [x_{ADPL} \quad x_{ATGR} \quad x_{LEDO} \quad x_{PODR}]'$ that minimize variance to a target expected return $\mu_{p,x} = \mu_{p,0}$.

If we want to find out the portfolio that has the same return as PODR we can use (31) and (33). In order to calculate variance we employ (11). The solution for this problem is:

$$x = \begin{bmatrix} 0.3484 \\ -0.1604 \\ 0.4459 \\ 0.3662 \end{bmatrix},\quad \mu_{p,x} = 0.012, \qquad \sigma_{p,x}^2 = 0.0025$$

This portfolio has the same average return as PODR but lower volatility. The volatility of this portfolio is 5%, while PODR volatility is 6.9% (see Table 1).

*Portfolio frontier*

Minimum variance portfolio, as well as previously calculated portfolio that has the same average return as PODR but lower volatility, are both the efficient portfolios and located on portfolio frontier. As shown by Black (1972) the portfolio frontier can be represented as convex combinations of any two frontier portfolios. If X and Y are two frontier portfolios, and if X ≠ Y, and let α be any constant, then the portfolio:

(33) $$z = \alpha \cdot x + (1 - \alpha) \cdot y$$

is also frontier portfolio with $\mu_{p,z} = z'\mu$ and $\sigma_{p,z}^2 = z'\Sigma z$.

The location of *z* portfolio on the frontier depends on α. If α=1, then portfolio *z* is equal to *x*, and if α=0, then portfolio *z* is equal to *y*. If α=1/2 then portfolio *z* is halfway between *x* and *y*. If α=2 we have two *x* portfolios and minus one *y* portfolio, and that is long-short portfolio. α can be any number.

We can also use α to calculate $\mu_{p,z}$ and $\sigma_{p,z}^2$:

(34) $$\mu_{p,z} = \alpha\mu_{p,x} + (1 - \alpha)\mu_{p,y}$$

(35) $\sigma^2_{p,z} = \alpha^2\sigma^2_{p,x} + (1-\alpha)^2\sigma^2_{p,y} + 2\alpha(1-\alpha)\sigma_{x,y}$

In previous calculations we reach two frontier portfolios. These are Minimum variance portfolio and portfolio that has the same return as PODR. Using (33), (34) and (35) we can calculate portfolio z, $\mu_{p,z}$ and $\sigma^2_{p,z}$ with arbitrary α=0.5:

$$z = 0.5\begin{bmatrix}0.291\\0.385\\0.288\\0.035\end{bmatrix} + 0.5\begin{bmatrix}0.3484\\-0.1604\\0.4459\\0.3662\end{bmatrix} = \begin{bmatrix}0.3197\\0.1126\\0.3670\\0.2006\end{bmatrix}$$

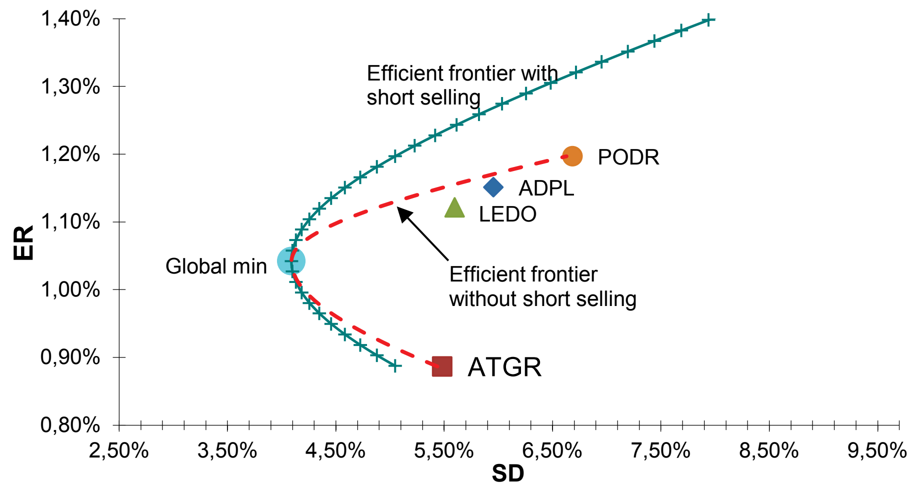
$$\mu_{p,z} = 0.5\cdot 0.01042 + 0.5\cdot 0.012 = 0.01121$$

$$\sigma^2_{p,z} = (0.5)^2(0.00168)^2 + (0.5)^2(0.0025)^2 + 2(0.5)(0.5)(0.0017) = 0.0019$$

Portfolio *z* lies on portfolio frontier between Minimum variance portfolio and portfolio that has the same return as PODR. Now, we can trace out all portfolios that lie on portfolio frontier by using various α's. The results are presented in Figure 3 and in table in appendix.

Figure 3

Portfolio frontier




*Source.* Author's calculation using software package EViews 7.0.

Notice that only portfolios that are located upper right from Minimum variance portfolio (including Minimum variance portfolio) are efficient portfolios. Anything below the global minimum variance portfolio is not efficient. Now we have to choose one of these efficient portfolios. All portfolios on efficient frontier have the smallest risk for a given target return. Which portfolio investors are going to choose depends on what their target expected return or what their target risk is. If investor is very risk averse, then he will choose portfolios very close to the global minimum variance portfolio. But, if investor is willing to take on higher risk, then he can get a higher average return if he chooses the portfolio from frontier that is placed in the upper right corner. This kind of portfolio is usually going to involve some short sales (negative assets), but in practice it is not possible to short assets on ZSE. If we impose restrictions on short selling ($x_i \geq 0$) then the efficient frontier with short selling is not feasible (Benninga, 2000). Only portfolios on dashed line at Figure 3 marked with text "Efficient frontier without short selling" are available. For a given level of risk, the short sale portfolio is usually more efficient (or at least equally efficient) because it has a higher expected return for the same level of risk.

## Conclusions

In order to construct portfolio with more than two assets based on modern portfolio theory it is practical to use matrix algebra. Presented example shows that, because of low correlation of underlying assets, it is possible to significantly reduce risks of investments by constructing the certain portfolio of chosen stocks from the Official Market segment of ZSE.

The Minimal variance portfolio has 29.1% AD Plastik, 38.5% Atlantic Grupa dd, 28.8% Ledo and 3.5% Podravka dd., expected monthly return 1.04%, and standard deviation 4.09%. If we compare these numbers with the return and standard deviation of each particular stock (see Table 1) we can see all benefits from diversification. Additionally, with matrix algebra and LaGrangian function it is possible to calculate portfolio that will have desired return. Portfolio with 34.84% of AD Plastik, -16.04% of Atlantic Grupa dd, 44.59% of Ledo and 36.62% of Podravka dd has the same average return as PODR but lower volatility. The volatility of this portfolio is 5%, while PODR volatility is 6.9%. Both Minimal variance portfolio and constructed portfolio with the same average return as PODR (and lower volatility) are the efficient portfolios and they lie on portfolio frontier.

In this paper the efficient portfolio frontier was calculated based on the fact that portfolio frontier can be represented as convex combination of any two frontier portfolios. Which portfolio investors are going to choose depends on what their target expected return is, or what their target risk is. Furthermore, when we impose the restrictions on short selling, the no short sale portfolio frontier lies "inside" of the frontier where we allow short sales. This significantly reduces the possibility for diversification and because of that we cannot construct such portfolio that will have higher return than the asset with the highest return in portfolio has.

Further analysis could be directed toward the calculation of the tangency portfolio for a given risk free rate. The tangency portfolio would be the efficient frontier portfolio that has the maximum Sharpe ratio. To determine that optimal portfolio empirically, the numerical calculation procedures and computer simulations (such as Monte Carlo method) could be used.

*References:*

*Benninga, S.* (2000). Financial Modelling (2nd ed.). Massachusetts Institute of Technology.

*Bertsekas, D. P.* (1999). Nonlinear Programming (2nd ed.). Cambridge, MA: Athena Scientific.

*Black, F.* (1972). Capital market equilibrium with restricted borrowing. - Journal of Business, Vol. 45, July, p. 444–455.

*Chen, N. F., R. Roll & S. A. Ross* (1986). Economic forces and the stock market. - Journal of business, p. 383-403.

*Chiang, A. C.* (1984). Fundamental Methods of Mathematical Economics (3rd ed.). McGraw-Hill.

*Dowdy, S., S. Wearden* (1983). Statistics for Research. Wiley.

*Elton J. E., J. M. Gruber* (2011). Investments and Portfolio Performance. World Scientific.

*Hung, S. Y., T. P. Liang, & V. W. C. Liu* (1996). Integrating arbitrage pricing theory and artificial neural networks to support portfolio management. - Decision support systems, 18(3), p. 301-316.

*Klein, R. W., & V. S. Bawa* (1976). The Effect of Estimation Risk on Optimal Portfolio Choice. - Journal of Financial Economics, 3(3), p. 215-231.

*Lee, C. F., J. E. Finnerty, D. H. Wort* (1990). Security Analysis and Portfolio Management. Auburn, WA, USA: Scott Foresman & Co.

*Lin, C., & P. J. Hsieh* (2004). A fuzzy decision support system for strategic portfolio management. - Decision Support Systems, 38(3), p. 383-398.

*Markowitz, H.* (1952). Portfolio Selection. - Journal of Finance, 7, p. 77-91.

*Markowitz, H.* (1991). Portfolio Selection: Efficient Diversification of Investments (2nd ed.). New York: Wiley, Cambridge, MA: Basil Blackwell.

*Parra, M. A., A. B. Terol, & M. R. Urıa* (2001). A fuzzy goal programming approach to portfolio selection. - European Journal of Operational Research, 133(2), p. 287-297.

*Ross, S. A.* (1976). The arbitrage theory of capital asset pricing. - Journal of economic theory, 13(3), p. 341-360.

*Sharpe, W. F.* (1963). A simplified model for portfolio analysis. - Management science, 9(2), p. 277-293.

*Tobin, J.* (1958). Liquidity preference as behavior towards risk. - The review of economic studies, p. 65-86.

*Van Horne, J. C., & J. M. Wachowicz* (2008). Fundamentals of financial management. Pearson Education.

*Zimmermann, H. G., R. Neuneier, & R. Grothmann* (2001). Active Portfolio-Management based on Error Correction Neural Networks. – In: NIPS, p. 1465-1472.

*Zivot, E.* (2002). Introduction to Computational Finance and Financial Econometrics. University of Washington, Department of Economics.

www.zse.hr - Zagreb Stock Exchange

*Appendix*

Frontier portfolios – short selling is aloud

| alpha | 1-alpha | E[Rp,z] | var(Rp,z) | SD(Rp,z) | ADPL, % | ATGR, % | LEDO, % | PODR, % |
|---|---|---|---|---|---|---|---|---|
| 2 | -1 | 0.0089 | 0.0025 | 0.0505 | 23 | 93 | 13 | -30 |
| 1.9 | -0.9 | 0.0090 | 0.0024 | 0.0488 | 24 | 88 | 15 | -26 |
| 1.8 | -0.8 | 0.0092 | 0.0022 | 0.0473 | 25 | 82 | 16 | -23 |
| 1.7 | -0.7 | 0.0093 | 0.0021 | 0.0458 | 25 | 77 | 18 | -20 |
| 1.6 | -0.6 | 0.0095 | 0.0020 | 0.0446 | 26 | 71 | 19 | -16 |
| 1.5 | -0.5 | 0.0096 | 0.0019 | 0.0435 | 26 | 66 | 21 | -13 |
| 1.4 | -0.4 | 0.0098 | 0.0018 | 0.0426 | 27 | 60 | 22 | -10 |
| 1.3 | -0.3 | 0.0100 | 0.0018 | 0.0418 | 27 | 55 | 24 | -6 |
| 1.2 | -0.2 | 0.0101 | 0.0017 | 0.0413 | 28 | 49 | 26 | -3 |
| 1.1 | -0.1 | 0.0103 | 0.0017 | 0.0410 | 29 | 44 | 27 | 0 |
| 1 | 0 | 0.0104 | 0.0017 | 0.0409 | 29 | 39 | 29 | 4 |
| 0.9 | 0.1 | 0.0106 | 0.0017 | 0.0410 | 30 | 33 | 30 | 7 |
| 0.8 | 0.2 | 0.0107 | 0.0017 | 0.0413 | 30 | 28 | 32 | 10 |
| 0.7 | 0.3 | 0.0109 | 0.0018 | 0.0418 | 31 | 22 | 34 | 13 |
| 0.6 | 0.4 | 0.0110 | 0.0018 | 0.0426 | 31 | 17 | 35 | 17 |
| 0.5 | 0.5 | 0.0112 | 0.0019 | 0.0435 | 32 | 11 | 37 | 20 |
| 0.4 | 0.6 | 0.0114 | 0.0020 | 0.0446 | 33 | 6 | 38 | 23 |
| 0.3 | 0.7 | 0.0115 | 0.0021 | 0.0458 | 33 | 0 | 40 | 27 |
| 0.2 | 0.8 | 0.0117 | 0.0022 | 0.0473 | 34 | -5 | 41 | 30 |
| 0.1 | 0.9 | 0.0118 | 0.0024 | 0.0488 | 34 | -11 | 43 | 33 |
| 0 | 1 | 0.0120 | 0.0025 | 0.0505 | 35 | -16 | 45 | 37 |
| -0.1 | 1.1 | 0.0121 | 0.0027 | 0.0523 | 35 | -22 | 46 | 40 |
| -0.2 | 1.2 | 0.0123 | 0.0029 | 0.0542 | 36 | -27 | 48 | 43 |
| -0.3 | 1.3 | 0.0124 | 0.0032 | 0.0561 | 37 | -32 | 49 | 47 |
| -0.4 | 1.4 | 0.0126 | 0.0034 | 0.0582 | 37 | -38 | 51 | 50 |
| -0.5 | 1.5 | 0.0127 | 0.0036 | 0.0604 | 38 | -43 | 52 | 53 |
| -0.6 | 1.6 | 0.0129 | 0.0039 | 0.0626 | 38 | -49 | 54 | 56 |
| -0.7 | 1.7 | 0.0131 | 0.0042 | 0.0648 | 39 | -54 | 56 | 60 |
| -0.8 | 1.8 | 0.0132 | 0.0045 | 0.0672 | 39 | -60 | 57 | 63 |
| -0.9 | 1.9 | 0.0134 | 0.0048 | 0.0695 | 40 | -65 | 59 | 66 |
| -1 | 2 | 0.0135 | 0.0052 | 0.0719 | 41 | -71 | 60 | 70 |
| -1.1 | 2.1 | 0.0137 | 0.0055 | 0.0744 | 41 | -76 | 62 | 73 |
| -1.2 | 2.2 | 0.0138 | 0.0059 | 0.0769 | 42 | -82 | 64 | 76 |
| -1.3 | 2.3 | 0.0140 | 0.0063 | 0.0794 | 42 | -87 | 65 | 80 |
| -1.4 | 2.4 | 0.0141 | 0.0067 | 0.0820 | 43 | -92 | 67 | 83 |
| -1.5 | 2.5 | 0.0143 | 0.0071 | 0.0845 | 43 | -98 | 68 | 86 |
| -1.6 | 2.6 | 0.0144 | 0.0076 | 0.0871 | 44 | -103 | 70 | 90 |
| -1.7 | 2.7 | 0.0146 | 0.0081 | 0.0898 | 45 | -109 | 71 | 93 |
| -1.8 | 2.8 | 0.0148 | 0.0085 | 0.0924 | 45 | -114 | 73 | 96 |
| -1.9 | 2.9 | 0.0149 | 0.0090 | 0.0951 | 46 | -120 | 75 | 99 |

*Note.* All of these portfolios lie on frontier portfolio, but only those with $\mu \geq \mu_{p,m}$ are efficient portfolios.

Frontier portfolios – short selling is not aloud

| alpha | 1-alpha | E[Rp,z] | var(Rp,z) | SD(Rp,z) | ADPL, % | ATGR, % | LEDO, % | PODR, % |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0.0089 | 0.00296 | 0.0544 | 0 | 100 | 0 | 0 |
| 0.1 | 0.9 | 0.009 | 0.00271 | 0.0521 | 3 | 94 | 3 | 0 |
| 0.2 | 0.8 | 0.0092 | 0.00249 | 0.0499 | 6 | 88 | 6 | 1 |
| 0.3 | 0.7 | 0.0093 | 0.0023 | 0.048 | 9 | 82 | 9 | 1 |
| 0.4 | 0.6 | 0.0095 | 0.00213 | 0.0462 | 12 | 75 | 12 | 1 |
| 0.5 | 0.5 | 0.0096 | 0.00199 | 0.0446 | 15 | 69 | 14 | 2 |
| 0.6 | 0.4 | 0.0098 | 0.00188 | 0.0433 | 17 | 63 | 17 | 2 |
| 0.7 | 0.3 | 0.01 | 0.00179 | 0.0423 | 20 | 57 | 20 | 2 |
| 0.8 | 0.2 | 0.0101 | 0.00172 | 0.0415 | 23 | 51 | 23 | 3 |
| 0.9 | 0.1 | 0.0103 | 0.00169 | 0.0411 | 26 | 45 | 26 | 3 |
| 1 | 0 | 0.0104 | 0.00167 | 0.0409 | 29 | 39 | 29 | 4 |
| 0.9 | 0.1 | 0.0106 | 0.0017 | 0.0412 | 26 | 35 | 26 | 13 |
| 0.8 | 0.2 | 0.0107 | 0.00178 | 0.0422 | 23 | 31 | 23 | 23 |
| 0.7 | 0.3 | 0.0109 | 0.00192 | 0.0438 | 20 | 27 | 20 | 32 |
| 0.6 | 0.4 | 0.011 | 0.00211 | 0.0459 | 17 | 23 | 17 | 42 |
| 0.5 | 0.5 | 0.0112 | 0.00235 | 0.0485 | 15 | 19 | 14 | 52 |
| 0.4 | 0.6 | 0.0114 | 0.00265 | 0.0515 | 12 | 15 | 12 | 61 |
| 0.3 | 0.7 | 0.0115 | 0.00301 | 0.0548 | 9 | 12 | 9 | 71 |
| 0.2 | 0.8 | 0.0117 | 0.00341 | 0.0584 | 6 | 8 | 6 | 81 |
| 0.1 | 0.9 | 0.0118 | 0.00388 | 0.0623 | 3 | 4 | 3 | 90 |
| 0 | 1 | 0.012 | 0.00439 | 0.0663 | 0 | 0 | 0 | 100 |

*Note.* The first part of the table is calculated as convex combinations of ATGR and global min. portfolio, and the second part are the convex combinations of global min. and PODR. Only portfolios with $\mu \geq \mu_{p,m}$ are efficient portfolios.

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